

Chapter 2 Laplace Transformation

$$L(f(t)) = F(s) \equiv \int_0^{\infty} e^{-st} f(t) dt, \quad f(t) = L^{-1}(F(s))$$

Properties:

$$1. \quad L(c_1 f(t) + c_2 g(t)) = c_1 L(f) + c_2 L(g)$$

$$2. \quad L(f') = sL(f) - f(0)$$

$$3. \quad L\left(\int_a^t f(t) dt\right) = \frac{1}{s} L(f) + \frac{1}{s} \int_a^0 f(t) dt$$

Pf: Let $G(t) = \int_a^t f(\tau) d\tau$, then $G'(t) = f(t)$

$$L(G') = sL(G) - G(0)$$

$$\rightarrow L(G) = \frac{1}{s} L(G') + \frac{1}{s} G(0) \rightarrow L\left(\int_a^t f(t) dt\right) = \frac{1}{s} L(f) + \frac{1}{s} \int_a^0 f(t) dt$$

$$4. \quad L(f'') = s^2 L(f) - sf(0) - f'(0)$$

$$L(f'') = sL(f') - f'(0) = s[sL(f) - f(0)] - f'(0) = s^2 L(f) - sf(0) - f'(0)$$

$$5. \quad L(f(at)) = \frac{1}{a} L(f) \Big|_{s/a}$$

$$\begin{aligned} L(f(at)) &= \int_0^{\infty} e^{-st} f(at) dt = \frac{1}{a} \int_0^{\infty} e^{-(s/a)\tau} f(\tau) d(\tau) = \frac{1}{a} \int_0^{\infty} e^{-(s/a)\tau} f(\tau) d\tau \\ &= \frac{1}{a} L(f) \Big|_{s/a} \end{aligned}$$

$$6. \quad L(e^{-at}) = 1/(s+a)$$

$$7. \quad L(\sin at, \cos at) = \frac{1}{s^2 + a^2} (a, s)$$

$$8. \quad L(t^n) = n! / s^{n+1}$$

$$9. \quad L(u(t)) = 1/s$$

$$10. \quad L(\cosh at, \sinh at) = (s, a) / (s^2 - a^2)$$

$$11. \quad L(e^{-at} f(t)) = L(f) \Big|_{s+a}$$

Ex. $y'' - 3y' + 2y = 12e^{-2t}$, $y(0) = 2$, $y'(0) = 6$

$$L(y'' - 3y' + 2y) = L(12e^{-2t}) = \frac{12}{s+2}$$

$$[s^2L(y) - y'(0) - sy(0)] - 3[sL(y) - y(0)] + 2L(y) = L(12e^{-2t}) = \frac{12}{s+2}$$

$$(s^2 - 3s + 2)L(y) = \frac{12}{s+2} + 2s, \quad L(y) = \frac{12}{(s+2)(s^2 - 3s + 2)} + \frac{2s}{(s^2 - 3s + 2)}$$

$$L(y) = \frac{12}{(s+2)(s-1)(s-2)} + \frac{2s}{(s-1)(s-2)} = \frac{a}{s-1} + \frac{b}{s-2} + \frac{c}{s+2}$$

$$a = (s-1)L(y)|_{s=1} = -6, \quad b = (s-2)L(y)|_{s=2} = 7, \quad c = (s+2)L(y)|_{s=-2} = 1$$

$$L(y) = -\frac{6}{s-1} + \frac{7}{s-2} + \frac{1}{s+2}, \quad y(t) = -6e^t + 7e^{2t} + e^{-2t}$$

Ex. $y' + 3y + 2\int_0^t y dt = e^{-3t}$, $y(0) = 1$

$$L(y' + 3y + 2\int_0^t y dt) = L(e^{-3t}) = \frac{1}{s+3}, \quad sL(y) - y(0) + 3L(y) + \frac{2}{s}L(y) = \frac{1}{s+3}$$

$$(s+3+\frac{2}{s})L(y) = \frac{1}{s+3} + 1,$$

$$L(y) = \frac{s}{(s+3)(s+1)(s+2)} + \frac{s}{(s+1)(s+2)} = \frac{a}{s+1} + \frac{b}{s+2} + \frac{c}{s+3}$$

$$a = (s+1)L(y)|_{s=-1} = -\frac{3}{2}, \quad b = (s+2)L(y)|_{s=-2} = 4, \quad c = (s+3)L(y)|_{s=-3} = -\frac{3}{2}$$

$$L(y) = -\frac{3}{2} \frac{1}{s+1} + 4 \frac{1}{s+2} - \frac{3}{2} \frac{1}{s+3}, \quad y(t) = -\frac{3}{2}e^{-t} + 4e^{-2t} - \frac{3}{2}e^{-3t}$$

Ex. $y' + 2y + 6\int_0^t z dt = -2u(t)$, $y' + z' + z = 0$, $y(0) = 1$, $z(0) = 1$

$$L(y' + 2y + 6\int_0^t z dt) = -2L(u(t)) \rightarrow s(s+2)L(y) + 6L(z) = -2 + s$$

$$L(y' + z' + z) = 0 \rightarrow sL(y) + (s+1)L(z) = 2$$

$$L(y) = \frac{\begin{vmatrix} -2+s & 6 \\ 2 & s+1 \end{vmatrix}}{\begin{vmatrix} s(s+2) & 6 \\ s & s+1 \end{vmatrix}} = \frac{s^2 - s - 14}{s(s+2)(s+1) - 6s} = \frac{s^2 - s - 14}{s(s+4)(s-1)}$$

$$= \frac{7}{2} \frac{1}{s} + \frac{3}{10} \frac{1}{s+4} - \frac{14}{5} \frac{1}{s-1},$$

$$y(t) = \frac{7}{2}u(t) + \frac{3}{10}e^{-4t} - \frac{14}{5}e^t$$

$$L(z) = \frac{\begin{vmatrix} s(s+2) & -2+s \\ s & 2 \end{vmatrix}}{\begin{vmatrix} s(s+2) & 6 \\ s & s+1 \end{vmatrix}} = \frac{s^2 + 6s}{s(s+4)(s-1)} = \frac{s+6}{(s+4)(s-1)} = \frac{7}{5} \frac{1}{s-1} - \frac{2}{5} \frac{1}{s+4}$$

$$z(t) = \frac{7}{5}e^t - \frac{2}{5}e^{-4t}.$$

Theorem.

If $f(t)$ is continuous on $(0, \infty)$, $\lim_{t \rightarrow 0^+} f(t)$ exists, $f'(t)$ is at least

piecewise regular on $[0, \infty)$ and both $f(t)$ and $f'(t)$ are of exponential

order, then

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{s \rightarrow \infty} sL(f) = f(0^+), \quad \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sL(f)$$

$$\text{Pf. } L(f') = sL(f) - f(0^+)$$

$$\lim_{s \rightarrow \infty} L(f') = \lim_{s \rightarrow \infty} \int_0^\infty e^{-st} f' dt = \lim_{s \rightarrow \infty} [sL(f) - f(0^+)] = 0 \rightarrow \lim_{s \rightarrow \infty} sL(f) = f(0^+)$$

$$\lim_{s \rightarrow 0} \int_0^\infty e^{-st} f' dt = \lim_{s \rightarrow 0} [sL(f) - f(0^+)] = \int_0^\infty f' dt = f|_0^\infty \rightarrow \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sL(f)$$

Theorem.

$$\text{If } L(f) = s\phi(s), \text{ then } f(t) = \frac{d}{dt} L^{-1}(\phi)$$

$$\text{Pf: Let } F(t) = L^{-1}(\phi), \text{ then } L(F) = \phi, \quad L(F') = s\phi - F(0)$$

$$\text{But, } F(0) = \lim_{s \rightarrow \infty} sL(F) = \lim_{s \rightarrow \infty} s\phi = 0, \text{ So, } L(F') = s\phi \rightarrow f(t) = F' = \frac{d}{dt} L^{-1}(\phi)$$

$$\text{Ex. } L^{-1}\left(\frac{s}{s^2+4}\right) = \frac{d}{dt} L^{-1}\left(\frac{1}{2} \frac{2}{s^2+4}\right) = \frac{1}{2} \frac{d}{dt} L^{-1}\left(\frac{2}{s^2+4}\right) = \frac{1}{2} \frac{d}{dt} \sin 2t = \cos 2t$$

Theorem.

$$\text{If } L(f) = \phi(s)/s, \text{ then } f(t) = \int_0^t L^{-1}(\phi(s)) dt$$

$$\text{Pf: Let } F(t) = L^{-1}(\phi), \text{ then } L\left(\int_0^t F dt\right) = \frac{1}{s} L(F) = \frac{\phi}{s} = L(f)$$

$$\rightarrow f = \int_0^t F dt = \int_0^t L^{-1}(\phi) dt = \frac{1}{s}$$

$$\begin{aligned} \text{Ex. } L^{-1}\left[\frac{1}{s(s^2+4)}\right] &= \int_0^t L^{-1}\left(\frac{1}{s^2+4}\right) dt = \frac{1}{2} \int_0^t L^{-1}\left(\frac{2}{s^2+4}\right) dt = \frac{1}{2} \int_0^t \sin 2t dt \\ &= \frac{1}{2} \frac{1}{2} (-\cos 2t) \Big|_0^t = \frac{1}{4} (1 - \cos 2t) \end{aligned}$$

Properties:

$$\text{a. } L(e^{-at} f) = L(f) \Big|_{s=s+a}, \text{ b, } L^{-1}(\phi) = e^{-at} L^{-1}(\phi(s-a))$$

$$\text{Ex. } L(e^{-at} \sin bt) = L(\sin bt) \Big|_{s=s+a} = \frac{b}{s^2 + b^2} \Big|_{s=s+a} = \frac{b}{(s+a)^2 + b^2}$$

$$\text{Ex. } L(e^{-at} \cos bt) = L(\cos bt) \Big|_{s=s+a} = \frac{s}{s^2 + b^2} \Big|_{s=s+a} = \frac{s+a}{(s+a)^2 + b^2}$$

$$\text{Ex. } L(e^{-at} t^n) = \frac{n!}{s^{n+1}} \Big|_{s=s+a} = \frac{n!}{(s+a)^{n+1}}$$

Ex.

$$\begin{aligned} L^{-1}\left(\frac{2s+5}{s^2+4s+13}\right) &= L^{-1}\left[\frac{2(s+2)+1}{(s+2)^2+3^2}\right] = L^{-1}\left[\frac{2s+1}{s^2+3^2}\right] \Big|_{s=s+2} \\ &= 2L^{-1}\left[\frac{s+1}{s^2+3^2}\right] \Big|_{s=s+2} + \frac{1}{3} L^{-1}\left[\frac{3}{s^2+3^2}\right] \Big|_{s=s+2} \\ &\rightarrow 2e^{-2t} \cos 3t + \frac{1}{3} e^{-2t} \sin 3t = e^{-2t} (2 \cos 3t + \frac{1}{3} \sin 3t) \end{aligned}$$

$$\text{Ex. } y'' + 2y' + y = te^{-t}, \quad y(0) = 1, \quad y'(0) = 2$$

$$\{s^2 L(y) - sy(0)\} - y'(0) + 2[sL(y) - y(0)] + L(y) = L(te^{-t}) = \frac{1}{(s+1)^2}$$

$$(s+1)^2 L(y) = \frac{1}{(s+1)^2} + s \rightarrow L(y) = \frac{1}{(s+1)^4} + \frac{s+1-1}{(s+1)^2} = \left[\frac{1}{s} - \frac{1}{s^2} + \frac{1}{3!} \frac{3!}{s^4}\right] \Big|_{s=s+1}$$

$$y = e^{-t} \left[u(t) - t + \frac{t^3}{3!} \right]$$

$$\text{Ex. } L[f(t-a)u(t-a)] = \int_0^\infty e^{-st} f(t-a)u(t-a) dt = e^{-as} \int_0^\infty e^{-s(t-a)} f(t-a)u(t-a) dt$$

Let $t-a = \tau \rightarrow$

$$L[f(t-a)u(t-a)] = e^{-as} \int_{-a}^\infty e^{-s\tau} f(\tau)u(\tau) d\tau = e^{-as} \int_0^\infty e^{-s\tau} f(\tau)u(\tau) d\tau = e^{-as} L(f)$$

$$\text{Ex. } L[f(t)u(t-a)] = e^{-as} L[f(t+a)]$$

Ex. If $L^{-1}(\phi) = f(t)$, then $L^{-1}[e^{-as}\phi] = f(t-a)u(t-a)$

Property: If $L^{-1}(\phi) = f(t)$, then $L(tf) = -\phi'(s)$

$$\phi(s) = \int_0^\infty e^{-st} f(t) dt \rightarrow -\phi'(s) = \int_0^\infty e^{-st} t f(t) dt = L(tf) \rightarrow$$

$$(-1)^2 \phi'''(s) = \int_0^\infty e^{-st} t^2 f(t) dt = L(t^2 f) \rightarrow (-1)^n \phi^{(n)}(s) = L(t^n f) \rightarrow$$

$$t^n f = (-1)^n L^{-1}[\phi^{(n)}(s)]$$

Ex. $L(t^2 \sin 2t) = (-1)^2 \frac{d^2}{ds^2} L(\sin 2t)$

Ex. $L(y) = \ln\left(\frac{s+1}{s-1}\right)$

$$-\frac{d}{ds} \ln\left(\frac{s+1}{s-1}\right) = -\frac{d}{ds} [\ln(s+1) - \ln(s-1)] = -\frac{1}{s+1} + \frac{1}{s-1} = L[e^t - e^{-t}]$$

$$ty = e^t - e^{-t} = \frac{\sinh t}{2}, \quad y = \frac{\sinh t}{2t}$$

Property: If $L(f) = \phi(s)$, then $L\left(\frac{f}{t}\right) = \int_s^\infty \phi ds$

Pf: $L(f) = \phi(s) = \int_0^\infty e^{-st} f dt \rightarrow$

$$\int_s^\infty \phi ds = \int_s^\infty \int_0^\infty e^{-st} f dt ds = \int_0^\infty f \int_s^\infty e^{-st} ds dt = \int_0^\infty f(t) \left(-\frac{e^{-st}}{t}\right) \Big|_s^\infty dt = \int_0^\infty e^{-st} \frac{f}{t} dt$$

$$= L\left(\frac{f}{t}\right) \rightarrow f = tL^{-1}\left(\int_s^\infty \phi ds\right)$$

Ex. $L\left(\frac{\sin at}{t}\right) = \int_s^\infty L(\sin at) ds = \int_s^\infty \frac{a}{s^2 + a^2} ds$

Let $s = a \tan \theta \rightarrow \theta = \tan^{-1}(s/a), \quad ds = a \sec^2 \theta d\theta,$

$$\int_{\tan^{-1}(s/a)}^{\pi/2} d\theta = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left(\frac{s}{a}\right)$$

Ex.

$$y = L^{-1}\left[\frac{s}{(s^2+1)^2}\right] \rightarrow L(y) = \frac{s}{(s^2+1)^2} \rightarrow$$

$$L\left(\frac{y}{t}\right) = \int_s^\infty \frac{s}{(s^2+1)^2} ds = \frac{1}{2} \int_s^\infty \frac{d(s^2+1)}{(s^2+1)^2} = -\frac{1}{2} \frac{1}{s^2+1} \Big|_s^\infty = \frac{1}{2} \frac{1}{s^2+1} = L\left(\frac{\sin t}{2}\right)$$

$$y = \frac{1}{2} t \sin t$$

Ex.

$$y = L^{-1}\left[\frac{s}{(s^2-1)^2}\right] \rightarrow L(y) = \frac{s}{(s^2-1)^2} \rightarrow$$

$$L\left(\frac{y}{t}\right) = \int_s^\infty \frac{s}{(s^2-1)^2} ds = \frac{1}{2} \int_s^\infty \frac{d(s^2-1)}{(s^2-1)^2} = -\frac{1}{2} \frac{1}{s^2-1} \Big|_s^\infty = \frac{1}{2} \frac{1}{s^2-1} = L\left(\frac{\sinh t}{2}\right)$$

$$y = \frac{1}{2} t \sinh t$$

Hw. Find the Laplace transformation of the following forms:

a) $t \int_0^t e^{-3t} \sin(2t) dt$, b) $e^{-3t} \int_0^t t \sin(2t) dt$, c) $\int_0^t t e^{-3t} \sin(2t) dt$, d) $\int_0^t e^{-3t} \frac{\sin(2t)}{t} dt$

e) $e^{-3t} \int_0^t \frac{\sin(2t)}{t} dt$

Hw. Find the inverse Laplace transformation of the following forms:

a) $\ln \frac{s+a}{s+b}$, b) $\frac{e^{-s} + e^{-2s}}{s^2 - 3s + 2}$, c) $s \ln\left(\frac{s-1}{s+1}\right)$, d) $\frac{1}{s} \tan^{-1} \frac{1}{s}$, e) $\ln \frac{s^2+1}{s(s+1)}$, f)

$$\frac{1}{(s+2)^4},$$

g) $\frac{s}{(s+2)^4}$, h) $\frac{1}{s(s+2)^4}$, i) $\frac{e^{-s}}{s(s+2)^4}$

$$\S \quad f(t) = L^{-1}[p(s)/q(s)]$$

Case 1. if $\frac{p(s)}{q(s)} = \sum_{i=1}^n \frac{a_i}{s-c_i}$, all c_i are distinct, then

$$f(t) = \sum_{i=1}^n a_i e^{c_i t}, \text{ where } a_i = (s-c_i) \frac{p(s)}{q(s)} \Big|_{s=c_i}$$

Case 2 if $\frac{p(s)}{q(s)} = \sum_{i=1}^k \frac{a_i}{s-c_i} + \frac{b_1}{s-d} + \dots + \frac{b_m}{(s-d)^m}$, all c_i are distinct, then

$$\frac{p(s)}{q(s)} = \sum_{i=1}^k \frac{a_i}{s-c_i} + \frac{b_1}{s-d} + \dots + b_m \frac{1}{(m-1)!} \frac{(m-1)!}{(s-d)^m},$$

$$y(t) = \sum_{i=1}^k a_i e^{c_i t} + e^{dt} \left(b_1 + \dots + b_m \frac{t^{m-1}}{(m-1)!} \right), \text{ where } b_m = \frac{1}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left[(s-d)^m \frac{p(s)}{q(s)} \right] \Big|_{s=d}$$

Case 3 if

$$\frac{p(s)}{q(s)} = \frac{As+B}{(s+a)^2+b^2} = \frac{A(s+a)+(B-Aa)}{(s+a)^2+b^2} = A \frac{(s+a)}{(s+a)^2+b^2} + \frac{B-Aa}{b} \frac{b}{(s+a)^2+b^2}$$

$$L^{-1}\left[\frac{p(s)}{q(s)}\right] = e^{-at}\left[A\cos t + \frac{B-Aa}{b}\sin t\right]$$

$$\text{Ex. } L^{-1}\left[\frac{s^2+2}{s(s+1)(s+2)}\right]$$

$$\begin{aligned}\frac{s^2-s+s+2}{s(s+1)(s+2)} &= \frac{s+1-2}{(s+1)(s+2)} + \frac{1}{s(s+1)} = \frac{1}{s+2} + \frac{-2}{(s+1)(s+2)} + \frac{1}{s(s+1)} \\ &= \frac{1}{s+2} - 2\left[\frac{1}{s+1} - \frac{1}{s+2}\right] + \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s} - \frac{3}{s+1} + \frac{3}{s+2}\end{aligned}$$

$$L^{-1}\left[\frac{s^2+2}{s(s+1)(s+2)}\right] = u(t) - 3e^{-t} + 3e^{-2t}$$

$$\text{Ex. } L^{-1}\left[\frac{s}{(s+2)^2(s^2+2s+10)}\right]$$

$$\frac{s}{(s+2)^2(s^2+2s+10)} = \frac{s}{(s+2)^2[(s+1)^2+3^2]} = \frac{a}{s+2} + \frac{b}{(s+2)^2} + \frac{c(s+1)+d}{[(s+1)^2+3^2]}$$

$$s = [a(s+2)+b][(s+1)^2+3^2] + (s+2)^2[c(s+1)+d]$$

$$-2 = 10b, \quad b = -1/5$$

$$0.2(s^2+7s+10) = a(s+2)[(s+1)^2+3^2] + (s+2)^2[c(s+1)+d]$$

$$0.2(s+5) = a[(s+1)^2+3^2] + (s+2)[c(s+1)+d],$$

$$0.06 = a, \quad 20 - 6(s+1) = 100[c(s+1)+d], \quad c = -0.06, \quad d = 0.2$$

$$L^{-1}\left[\frac{s}{(s+2)^2(s^2+2s+10)}\right] = \left(\frac{3}{50} - \frac{1}{5}t\right)e^{-2t} + e^{-t}\left(\frac{1}{15}\sin t - \frac{3}{50}\cos t\right)$$

$$\text{Hw. a) } L^{-1}\left[\frac{s+1}{(s^2+1)(s^2+4s+13)}\right], \text{ b) } L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right], \text{ c) } L^{-1}\left[\frac{s^2-s+3}{s^3+6s^2+11s+6}\right]$$

$$\text{d) } L^{-1}\left[\frac{s+2}{s^4+4s^3+4s^2-4s-5}\right]$$

$$\text{Ex. } y''' + 3y'' + 3y' + y = e^{-t}, \quad y(0) = y'(0) = y''(0) = 0$$

$$L(y''' + 3y'' + 3y' + y) = (s^3 + 3s^2 + 3s + 1)L(y) = \frac{1}{s+1}$$

$$L(y) = \frac{1}{(s+1)^4} = \frac{1}{3!} \frac{3!}{(s+1)^4}, \quad y(t) = \frac{1}{3!} e^{-t} t^3$$

$$\text{Hw a) } y'' + 4y' + 4y = (t-2)e^{-(t-2)}u(t-2), \quad y(0) = 1, \quad y'(0) = -1$$

$$\text{b) } y'' + 4y' + 13y = \frac{1}{3}e^{-2t} \sin(3t), \quad y(0) = 1, \quad y'(0) = -2$$

§ Convolution $L[\int_0^t f(\tau)g(t-\tau)d\tau]$

$$\begin{aligned} L[\int_0^t f(\tau)g(t-\tau)d\tau] &= \int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau)d\tau dt \\ &= \int_0^\infty e^{-st} \int_0^t f(\tau)g(t-\tau)u(t-\tau)d\tau dt = \int_0^\infty e^{-st} \int_0^\infty f(\tau)g(t-\tau)u(t-\tau)d\tau dt \\ &= \int_0^\infty f(\tau) \int_0^\infty e^{-st} g(t-\tau)u(t-\tau)dt d\tau = \int_0^\infty e^{-s\tau} f(\tau)L(g)dt = L(f)L(g) \end{aligned}$$

$$\text{Ex. } y'' + 4y' + 13y = e^{-2t} \cos 3t, \quad y(0) = 1, \quad y'(0) = -1$$

$$L(y'' + 4y' + 13y) = L(e^{-2t} \cos 3t),$$

$$(s^2 + 4s + 13)L(y) = L(e^{-2t} \cos 3t) + s + 5$$

$$L(y) = \frac{1}{(s+2)^2 + 3^2} L(e^{-2t} \cos 3t) + \frac{s+2+3}{(s+2)^2 + 3^2}$$

$$y(t) = e^{-2t}(\cos 3t + \sin 3t) + \frac{1}{3} \int_0^t e^{-2\tau} \sin 3\tau e^{-2(t-\tau)} \cos 3(t-\tau) d\tau$$

$$y(t) = e^{-2t}(\cos 3t + \sin 3t) + \frac{1}{3} \int_0^t e^{-2\tau} \sin 3\tau \cos 3(t-\tau) d\tau$$

$$\text{Ex. } y'' + 2ay' + (a^2 + b^2)y = f(t), \quad y(0) = y'(0) = 0,$$

$$L[y'' + 2ay' + (a^2 + b^2)y] = L[f(t)]$$

$$[(s+a)^2 + b^2]L(y) = L[f(t)], \quad L(y) = \frac{1}{[(s+a)^2 + b^2]} L[f(t)]$$

$$y(t) = \frac{1}{b} \int_0^t e^{-a\tau} \sin(b\tau) f(t-\tau) d\tau$$

$$\text{Hw } y'' + 2ay' + (a^2 + b^2)y = f(t), \quad y(0) = c, y'(0) = d.$$

$$\text{Properties: } L\left(\frac{f}{t}\right) = \int_s^\infty L(f)ds, \quad L(tf) = \left(-\frac{d}{ds}\right)L(f), \quad L(t^n f) = \left(-\frac{d}{ds}\right)^n L(f)$$

$$\text{Properties; } \lim_{s \rightarrow \infty} sL(f) = f(0), \quad \lim_{s \rightarrow 0} sL(f) = \lim_{t \rightarrow \infty} f(t)$$

$$\text{Ex. } y'' + 2ty' - 4y = 1, \quad y(0) = y'(0) = 0$$

$$L(y'') = sL(y') - y'(0) = sL(y') = s[sL(y) - y(0)] = s^2 L(y)$$

$$L(ty') = -\frac{d}{ds}L(y') = -\frac{d}{ds}[sL(y) - y(0)] = -L(y) - s\frac{dL(y)}{ds}$$

$$L(y'' + 2ty' - 4y) = L(1) \rightarrow (s^2 - 6)L(y) - 2s\frac{dL(y)}{ds} = \frac{1}{s} \rightarrow \frac{dL(y)}{ds} + \frac{6-s^2}{2s}L(y) = -\frac{1}{2s^2}$$

Integrating factor

$$\mu(s) = \exp\left[\int \frac{6-s^2}{2s} ds\right] = \exp\left(3\ln s - \frac{s^2}{4}\right) = \exp(\ln s^3 - \frac{s^2}{4}) = s^3 \exp(-\frac{s^2}{4})$$

$$\frac{d}{ds}[L(y)s^3 e^{-s^2/4}] = -\frac{1}{2s^2} s^3 e^{-s^2/4} = -\frac{1}{2} s e^{-s^2/4}$$

$$L(y)s^3 e^{-s^2/4} = c - \frac{1}{2} \int s e^{-s^2/4} ds = c - \int e^{-s^2/4} d(s^2/4) = c + e^{-s^2/4}$$

$$L(y) = \frac{c}{s^3} e^{s^2/4} + \frac{1}{s^3} \quad \text{according to} \quad \lim_{s \rightarrow \infty} sL(y) = y(0) (=0) \quad \text{yields} \quad c = 0$$

$$L(y) = \frac{1}{s^3} = \frac{1}{2} \frac{2}{s^3}, \quad y(t) = \frac{1}{2} t^2$$

Ex. $ty'' + (4t-2)y' - 4y = 0, \quad y(0) = 1$

$$L(ty') = -\frac{d}{ds}L(y') = -\frac{d}{ds}\{sL(y) - y(0)\} = -L(y) - s\frac{d}{ds}L(y)$$

$$L(ty'') = -\frac{d}{ds}L(y'') = -\frac{d}{ds}\{sL(y') - y'(0)\} = -\frac{d}{ds}\{s[sL(y) - y(0)]\}$$

$$= -\frac{d}{ds}\{s^2 L(y) - s\} = -2sL(y) - s^2 \frac{d}{ds}L(y) + 1$$

$$ty'' + (4t-2)y' - 4y = 0 \rightarrow$$

$$[-2sL(y) - s^2 \frac{d}{ds}L(y) + 1] - 4[s \frac{d}{ds}L(y) + L(y)] - 2[sL(y) - 1] - 4L(y) = 0$$

$$(s^2 + 4s) \frac{d}{ds}L(y) + (4s + 8)L(y) = 3 \rightarrow \frac{d}{ds}L(y) + \frac{4s+8}{s^2+4s}L(y) = \frac{3}{s^2+4s}$$

Integrating factor

$$\mu(s) = \exp\left[\int \frac{4s+8}{s^2+4s} ds\right] = \exp\left[2\int \frac{2s+4}{s^2+4s} ds\right] = \exp[2\ln(s^2+4s)] = (s^2+4s)^2$$

$$\frac{d}{ds}[(s^2+4s)^2 L(y)] = 3(s^2+4s) \rightarrow s^2(s+4)^2 L(y) = c + s^3 + 6s^2$$

$$L(y) = \frac{c}{s^2(s+4)^2} + \frac{1}{s+4} + \frac{2}{(s+4)^2}, \quad \lim_{s \rightarrow \infty} sL(y) = \lim_{s \rightarrow \infty} \frac{s^2}{(s+4)^2} = 1 \quad \text{ok}$$

$$\frac{1}{s^2(s+4)^2} = \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s+4} + \frac{a_4}{(s+4)^2},$$

$$1 = (s+4)^2(a_1s + a_2) + s^2[a_3(s+4) + a_4], \quad a_4 = 1/16$$

$$-\frac{1}{16}(s-4) = (s+4)(a_1s+a_2) + s^2a_3, \quad a_3 = \frac{1}{32}, \quad a_2 = 1/16, \quad a_1 = -\frac{1}{32}$$

$$\frac{c}{s^2(s+4)^2} = c\left[\frac{1}{32}\frac{1}{s} + \frac{1}{16}\frac{1}{s^2} - \frac{1}{32}\frac{1}{s+4} + \frac{1}{16}\frac{1}{(s+4)^2}\right]$$

$$y = \frac{c}{32}[1 + 2t - e^{-4t} + 2te^{-4t}] + e^{-4t}(1 + 2t)$$

Ex. $y'' + 8ty' - 8y = 0, \quad y(0) = 0, \quad y'(0) = -4$

Let $Y(s) = L(y)$

$$L(ty') = -\frac{d}{ds}L(y') = -\frac{d}{ds}\{sY - y(0)\} = -\frac{d}{ds}(sY) = -s\frac{dY}{ds} - Y$$

$$L(y'') = sL(y') - y'(0) = s^2Y + 4$$

$$L(y'' + 8ty' - 8y) = 0 \rightarrow s^2Y + 4 - 8\left(s\frac{dY}{ds} + Y\right) - 8y = 0$$

$$8s\frac{dY}{ds} - (s^2 - 16)Y = 4, \quad \frac{dY}{ds} - \frac{s^2 - 16}{8s}Y = \frac{1}{2s}$$

Integrating factor $\mu(s) = \exp\left\{-\int \frac{s^2 - 16}{8s} ds\right\} = \exp\left\{-\frac{s^2}{16} + 2 \ln s\right\} = s^2 e^{-s^2/16}$

$$\frac{d}{ds}\{s^2 e^{-s^2/16} Y\} = \frac{1}{2} s e^{-s^2/16}, \quad s^2 e^{-s^2/16} Y = C + \frac{1}{2} \int s e^{-s^2/16} ds = C - 4e^{-s^2/16}$$

$$Y = C s^{-2} e^{s^2/16} - 4s^{-2}, \quad \lim_{s \rightarrow \infty} sY = 0 \text{ yields } C = 0, \quad Y = -4\frac{1}{s^2}, \quad y(t) = -4t$$

Ex. $ty'' + (t-1)y' + y = 0, \quad y(0) = 0$